

ADDITIONAL PROBLEMS: A COLLECTION OF PROBLEMS ABOUT LIGHT RAYS, REFRACTION AND RAINBOWS

Introduction

A rainbow is one of the most beautiful of all physical phenomena. They arise when, with the Sun behind you, light rays are returned by rain drops to your eyes; figure 1.

Why do rays return in such well-defined arcs to your eye, and what is that angle?

Why do rays of different colour subtend slightly different angles to your eye?

As is typical in physics, we break the question into sub-questions [each requiring its own physics and maths].



Figure 1: Mark Seton: “Double Rainbow over Doctors Pond”

- What is the trajectory of the light, from the Sun via the interior of the drop, to your eye? [Snell’s law of refraction, simple Greek geometry.]
- Why is there a bright bow at all when light from a wide variety of angles makes its way into our eyes? [Caustics, calculus to find maxima.]
- What is the angular extent of a rainbow? How do doubles arise and what is their angle? What about droplets of other than water? How wide and how thick are rainbows?

Rays and geometry

Consider a spherical droplet (refractive index n) with parallel rays from the Sun incident on it. Depending on where the beam strikes the sphere, the angle of incidence α varies. A beam that refracts into the drop, internally reflects and then refracts out; see figure 2 on which some radii have been drawn in.

- Find the angle χ the exit beam makes with the direction of the incident beam in terms of angles α and β . [Hint: find angles in the drop. Add deflections of the beam at A, B & C; figure 4 helps for this drawing.]
- Draw your own diagram which also shows the Sun (or at least its direction), droplets in a range of positions, your eye, and the subtended angle χ of the rainbow.
- Show that there is a maximum angle χ_m at a corresponding angle α_m . [Hint: use the chain rule when differentiating β since Snell’s law relates it to the angle of incidence α , that is $\sin(\beta) = \sin(\alpha)/n$.]
- Give a formula for χ_m in terms of n . What is χ_m for water ($n = 4/3$)?
- Write $\chi(\alpha)$ purely as a function of α , since Snell’s law eliminates β . Figure 3(a) is a plot of $\chi(\alpha)$ against α .

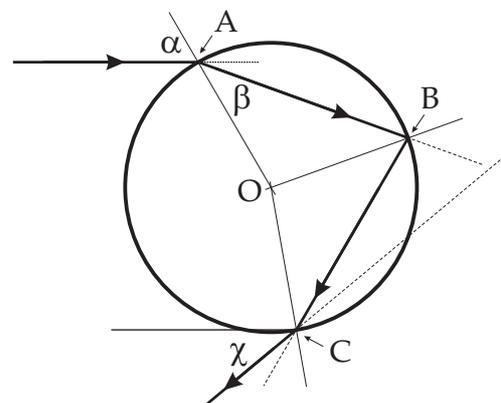


Figure 2: A ray passing through a droplet.

Why is the rainbow bright?

Refraction at A and C is accompanied by some external/internal reflection. Internal reflection at B has associated refraction of light out of the drop. The weight of reflection versus refraction depends on the “Fresnel Coefficients”. They are functions of α at A and of β (and hence, through Snell’s law, actually of α) at B and C. The emergent light is the intensity $H(\chi)$ times the angular interval $d\chi$ at C, that is $H(\chi) d\chi$,

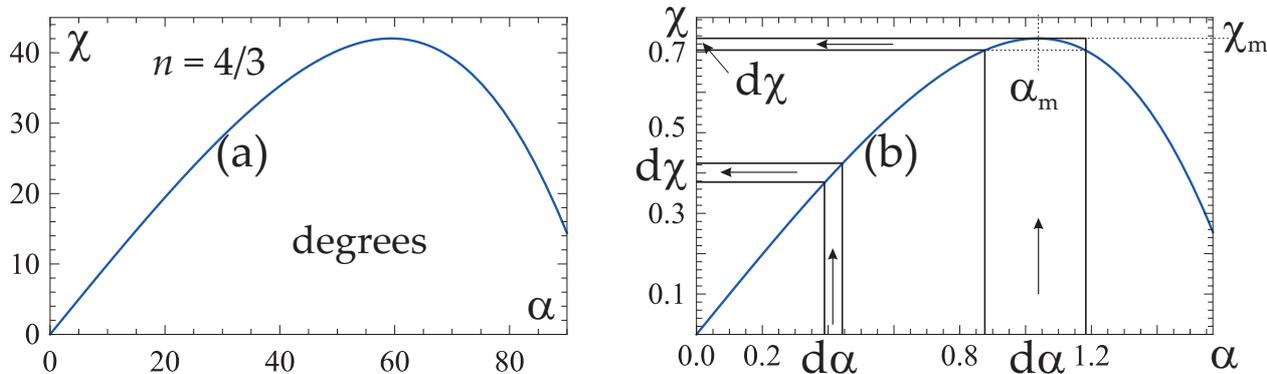


Figure 3: (a) Deflection $\chi(\alpha)$ against incidence α (degrees) for water. (b) Deflection $\chi(\alpha)$ against incidence α (radians) for water. Angular intervals of α are mapped into those of χ . The ratio $d\alpha/d\chi$ becomes critical at maximal deflection.

and depends on the intensity initially striking the drop at A, in the interval $d\alpha$, multiplied successively by the three relevant Fresnel Coefficients, compounded together as $I(\alpha) d\alpha$ say. These expressions are equal:

$$H(\chi) d\chi = I(\alpha) d\alpha \quad (1)$$

The angular distribution of intensity $H(\chi)$ that we see is very different from the input $I(\alpha)$. The ratio $d\alpha/d\chi$ behaves critically around (χ_m, α_m) : The intensity in an interval $(\alpha, \alpha + d\alpha)$ has to be mapped into a much smaller interval $(\chi, \chi + d\chi)$, that is, the light is intensified; see figure 3(b) & A-(c).

- (a) Argue in detail for why there are rainbows and that their angle subtended to the eye is χ_m . Qualitatively discuss why the sky under a rainbow is brighter than that above.
- (b) What happens as the refractive index of a liquid (i) vanishes, $n \rightarrow 1$ and (ii) gets very high (quantify “high”; these are toxic, arsenic-based liquids — do not try at home!)?

Colours, shape, other liquids, orders

- (a) A fit for how the refractive index of water changes with wavelength λ of light in the visible region is:

$$n(\lambda) = 1.31848 + 6.662/(\lambda[\text{nm}] - 129.2) \quad (2)$$

where λ is given in nanometers (blue ≈ 450 , green ≈ 530 , red ≈ 630). The variation $n(\lambda)$ is called “dispersion”. Thus different wavelengths of light have different angles χ_m where they are brightest, and rainbows get their colour.

- (i) Estimate the angular width of a rainbow.
 - (ii) Why is a rainbow darkest just outside the blue band?
- (b) We have considered a drop in the sky at an angle χ_m to the direction of the Sun’s rays. Drops anywhere on a circle subtending χ_m to the eye with the Sun shining perpendicular to the circle will contribute equally (hence the “bow”). But some drops might be below the horizon!
 - (i) Discuss, with diagrams, what fraction of a complete bow one can see at different times of day.
 - (ii) What ways are there of seeing complete rainbows?
 - (c) On the moon Titan of Jupiter it rains liquid methane ($n = 1.286$). What is the rainbow angle there?
 - (d) If it were to rain liquid hydrogen (with a very low refractive index $n = 1.0974$), what qualitative features would the rainbow angle have? Calculate what you would observe.
 - (e) Why are some rainbows brighter than others?
 - (f) How does the higher order bow arise? [Hint: you have to re-do the considerations of figure 2 from scratch; see figure 5 .] Calculate its subtended angle? Discuss why it is not as bright as the first order?
 - (g) What was the elevation of the Sun in the sky when Mark Seton took his photo (see Figure 1)? [Hint: the secondary rainbow helps.]

[MW; April 2014]

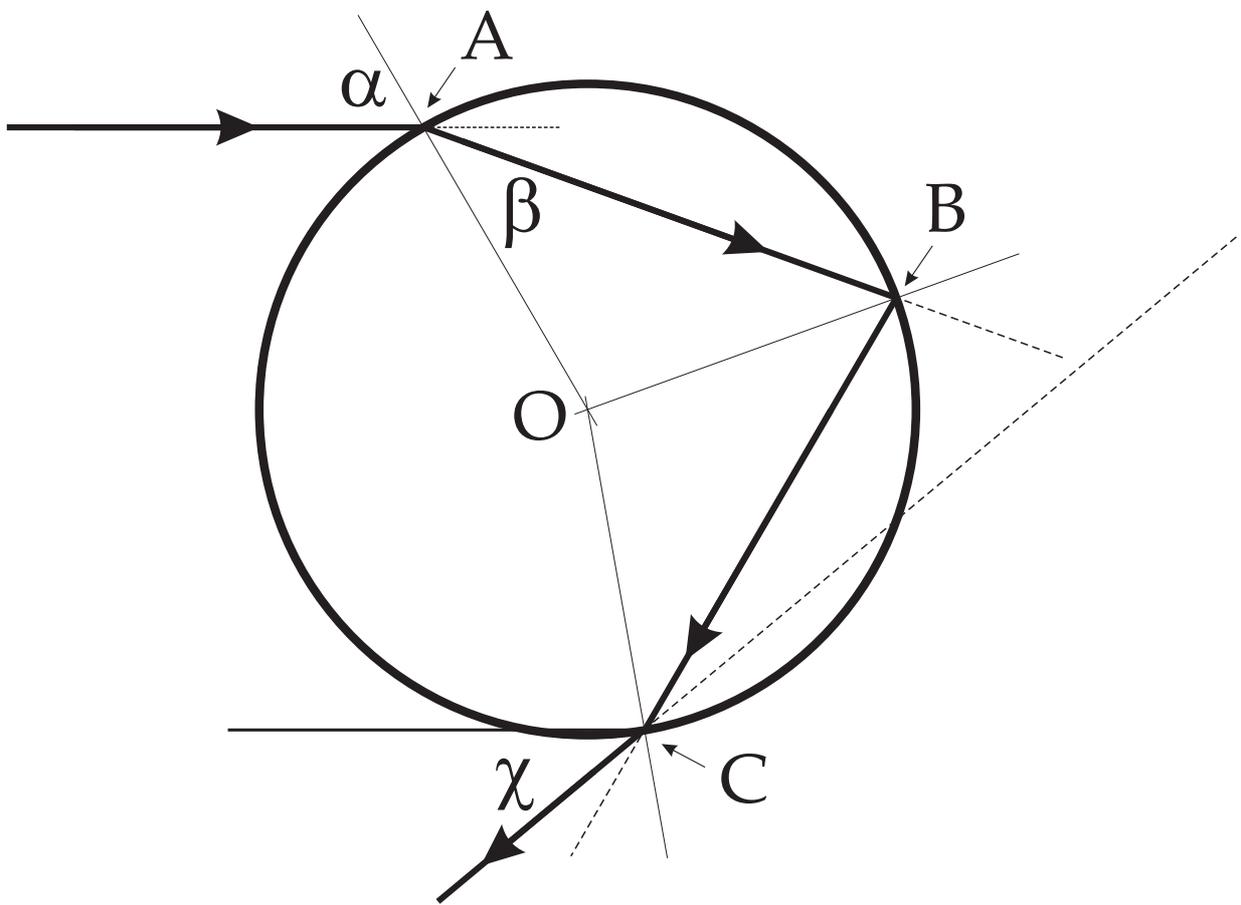


Figure 4: An enlarged figure 2 to use when answering section A.

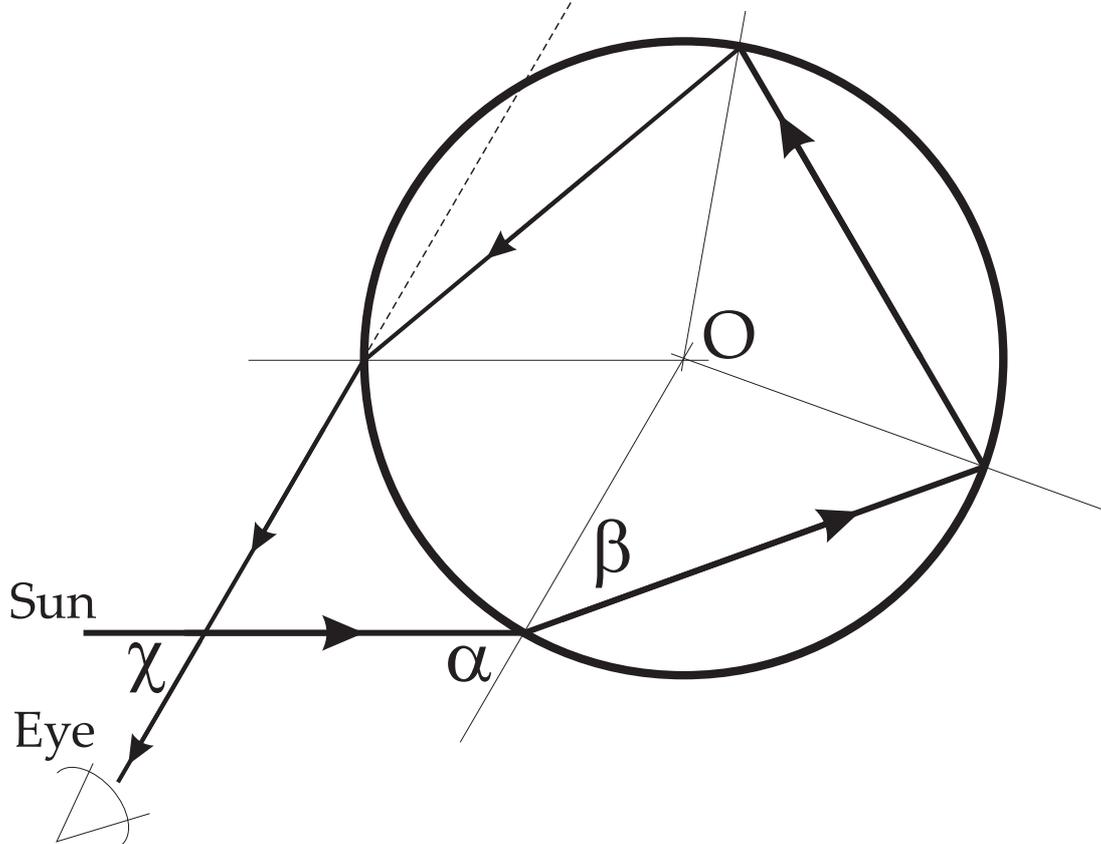


Figure 5: A sketch figure useful for answering C - (f).