ADDITIONAL PROBLEMS: **ON HANGING CHAINS, CHAIN FOUNTAINS, ARCHES, BRIDGES, . . .**

Introduction

These questions start with simple AS level statics and escalate slowly to A2 and beyond. The skills that are practised here are those that you would have to use as second nature if you go on to do problems in astrophysics, high energy physics, cosmology, . . .

There are still remarkable and unsolved problems in classical mechanics. Have you seen Steve Mould's chain fountain at `http://stevemould.com/siphoning-beads/`? Watch the chain climb out of the pot, apparently defying momentum conservation; see also figure 1. This collection of problems eventually explains phenomenon, which is really a research problem[1]. On the way, we deal with chain statics, dynamics, arches, and bridges, all problems with common mathematical and physical elements.
We urge you to try all these problems. Remember, only by doing the problems yourself will you take active control of your own physics. If you get really stuck, you can email the RPP Project for a pdf of hints.

Figure 1: Steve Mould holding a tub of chain aloft, demonstrating a fountain effect as the chain falls to the ground.

The nature of tension in a chain or rope

Why, if there are forces in a rope (being pulled by forces $\underline{f}$ and $-\underline{f}$ at its ends), do the parts of the rope not accelerate? The internal system of forces are actually "tensions". At each point they act equally but oppositely to put the body under tension; see the boxed detail in figure 2 where a rope is shown in tension from the forces acting at each of its ends.
Consider the central slice $\mathrm{d}x$ of the rope. If $T(x) = T(x + \mathrm{d}x)$ there is no net force on the slice and it remains unaccelerated. For slices at the ends of the rope there is a tension acting on the internal side of the slices, but not at the external sides. On these slices, external forces f are needed so there is a force balance for each slice and they don't accelerate. For this to hold, clearly the magnitudes $f = T$. The fs are forces, but it is they that create the tensions in the rope/chain.

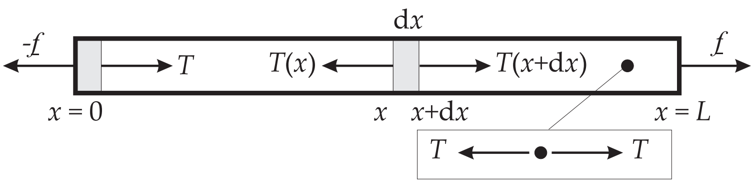

Figure 2: Tensions in a rope being pulled at its ends.

Tensions when hanging under gravity

A chain hanging from one end doesn't have tensions that are constant along its length. The difference now between $T(x)$ and $T(x + \mathrm{d}x)$ acting on a slice is needed to support the weight $(\lambda\,\mathrm{d}x)g$. Here λ is the mass per unit length of the rope and hence $(\lambda\,\mathrm{d}x)$ is the slice's mass. Multiplying by g gives the slice's weight.

[1] See JS Biggins and M Warner, Proceedings of the Royal Society A, 2014. `http://dx.doi.org/10.1098/rspa.2013.0689` also (free) at `http://arxiv.org/pdf/1310.4056v2.pdf`.
A manual for making a chain fountain and performing experiments with it is available at `http://www.tcm.phy.cam.ac.uk/~mw141/chain_fountain_manual.pdf`.

Draw a diagram similar to figure 2, but vertical, to get a feeling for the forces in a chain hanging from one end, where $x = 0$.
Exercise 1: A chain of total length L and mass λ per unit length is hanging vertically from one end where a supporting force f is applied to stop the chain falling. By considering the mass of chain below x, give an expression for the tension $T(x)$, where x is measured from $x = 0$ at the upper end of the chain. [MW]

An alternative method involves differentials — do this if you have met Taylor expansions[2] and differential equations. The difference in tensions supporting the weight can be expressed in the form:

$$T(x) - T(x + \mathrm{d}x) = \lambda g \, \mathrm{d}x.$$

From this relation one could derive a simple differential equation for $T(x)$ with a simple solution.

The nature of chains, ropes, masonry, arches

Chains, ropes, threads etc. share the property that their internal forces (tensions) must be along their lengths — tangential to their curves, to be mathematically precise. They have negligible resistance to bend, at least at long enough length scales, so that if there were a sideways component to the forces within them, they would deflect until forces were again tangential.
Masonry (stone, brick, . . .) has a similar character, but in reverse. It is strong in compression, but weak in extension or shear (deflection). The pairs of forces T seen in a rope are now reversed to be in compression.

A chain supported at both ends forms a classical shape called a *catenary* (first studied by Hooke in the 1670s, and described mathematically by Leibniz, Huygens and Johann Bernoulli in 1691); see fig. 3. The chain's own weight is supported by a system of tensions, varying along the chain, that are tangential. A masonry arch has to support its own weight by a system of tangential compressions. Take the shape of a hanging chain: reverse gravity to $-\underline{g}$ (stand on your head!), and you have the ideal arch — pointing upwards and the tensions reversed from those of the downward pointing chain). This technique of determining arches goes back to the Middle Ages; wax on a hanging thread solidifies it, allowing it to be turned over as an arch with tangential compressions.

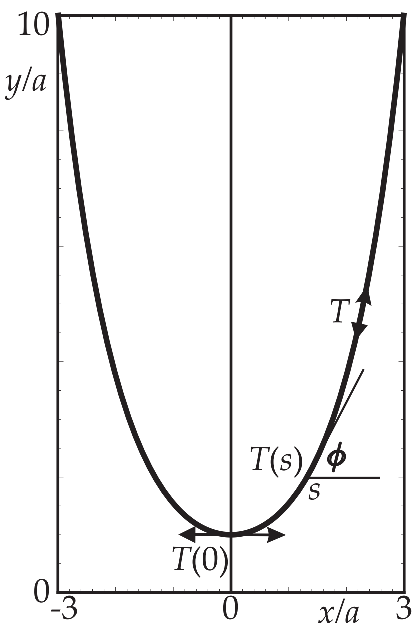

Figure 3: A catenary formed by a chain supported at its two ends. Ex. 7 shows it has the form $y/a = \cosh(x/a)$. The angle between the tangent and the x axis is ϕ. Tensions $T(s)$ are shown, in particular at $s = 0$.

The shape of a hanging chain – 1

Exercise 2: Consider a chain supported at its ends and hanging under gravity. Let the mass per unit length be λ and measure lengths s along the chain from its mid-point (the lowest point, since we consider the case where the end supports are at the same height). Let $T(s)$ be the tension at s along the chain. Show that the shape it adopts, the catenary, has:

$$\tan \phi(s) = \frac{s}{a}$$

where $\phi(s)$ is the angle that the tangent to the chain at a distance s makes to the horizontal, and a is a length you should define in terms of $T(0)$, λ and g.
This form is known as the "intrinsic form" of the curve (as against, for instance, the *Cartesian* form).
Hint: Consider the static equilibrium of a section of chain extending from $s = 0$ to the point s. The tension $T(0)$ at $s = 0$ is horizontal, with $T(s)$ at s having an angle $\phi(s)$ to the horizontal. The horizontally and vertically resolved tension and weight forces must balance. The mass between the lowest point and s is λs,

[2] $T(x + \mathrm{d}x) = T(x) + \frac{\mathrm{d}T}{\mathrm{d}x}\,\mathrm{d}x + \ldots$; see the relevant Maths concepts pages.

the weight of which of course acts vertically.

MW & PS.

The dynamics of chains

When a chain is pulled into motion by a force applied at one end, both momentum and kinetic energy are created. It is deeply worrying that these turn out to be at variance with each other, as the problem below illustrates. First some simple book keeping for energy and momentum for distributed objects like chains.

A length s of chain has mass λs. If the speed is v, then the momentum is $\lambda s v$ and the kinetic energy is $\frac{1}{2}\lambda s v^2$. If a chain is being dragged out of a pile of stationary chain with speed v, then the length s already out of the pile and in motion increases at a rate $\mathrm{d}s/\mathrm{d}t = v$. Thus the rate of momentum change is $\mathrm{d}(\lambda s v)/\mathrm{d}t = \lambda v\,\mathrm{d}s/\mathrm{d}t = \lambda v^2$. Likewise the rate of change of kinetic energy is $\frac{1}{2}\lambda v^3$ (derive this!).

Exercise 3: Picking up a chain at constant speed, v, neglecting gravity. Show that a force λv^2 must be exerted on the end of a chain to start setting it in motion. What power is exerted by the force? Compare this with the rate of creation of kinetic energy. Comment!

Hint: Force is the rate of change of momentum.

MW

Exercise 4: Consider a long length of chain in a pile on a flat surface. One end is raised against gravity g at a constant speed v by applying an upward force $f = T(0)$, the tension at the end of the chain. Show that when a length L of chain has been raised, then $T(0) = \lambda L g + \lambda v^2$. Further determine the power input by f and compare this with the rate of change of stored energy in the chain. [*MW*]

Discussion of energy and momentum in distributed objects

The $\frac{1}{2}$ in KE $= \frac{1}{2}\lambda v^2$ is not consistent with the momentum flux[3] expression λv^2. One half of the energy is lost during start up of the motion by any force that is doing work while providing momentum. It is the same in many such problems in mechanics, for instance when ore or coal is put on a moving conveyor belt. We also see it in other branches of physics, for instance connecting a capacitance C across a battery delivering charge at a potential V takes a charge $Q = CV$ from the battery at potential V. Thus the battery delivers work $QV = CV^2$. But the energy stored in the capacitor when it fully charged is only $\frac{1}{2}CV^2$. The problem is that charge is delivered to the capacitor through a finite potential difference from V to the current potential of the capacitor. (The difference gets ever smaller as the capacitor charges up.) That portion of the energy is not storable, something that is avoided in AC circuits where the potential varies with time and matches that across the capacitor as charge flows in.

The motion of chains and ropes around corners

The motion of ropes along their own length is most surprising. It takes us a step closer to the chain fountain problem. It was much studied in the 1850s (including by the Astronomer Royal) since there was industrial interest in laying submarine cables for telecommunications. The problems found their way into the Cambridge exams certainly by 1854 and then by 1860 into main-stream mechanics text books, for instance that of Routh.

Consider a rope of mass/unit length λ under tension T going around a pulley at a speed v; see figure 4. Let us show that if the speed and tension are appropriately related, then one does not need the pulley at all! [Ignore gravity for the moment.] Results for centripetal motion are required [see the RSPP concept page on circular motion, and the relevant questions] and we sketch them here.

In circular motion, the inward radial acceleration is v^2/r, with r the circle's (here, the pulley's) radius; see figure 5. A length of arc $\mathrm{d}s$ has mass $\lambda\,\mathrm{d}s$ and hence a force $(\lambda v^2/r)\,\mathrm{d}s$ is required (from "$f = ma$") to accelerate the element inwards as it goes around the arc. The effects of tensions in a circular arc of rope are similar to the effects of a tangential v leading to v^2/r. The two forces $T(s)$ and $T' = T(s + \mathrm{d}s)$ in figure 5

[3] See Flux in the Maths concepts pages on vectors.

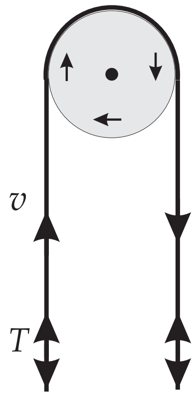

Figure 4: Rope under tension T winding around a pulley at speed v.

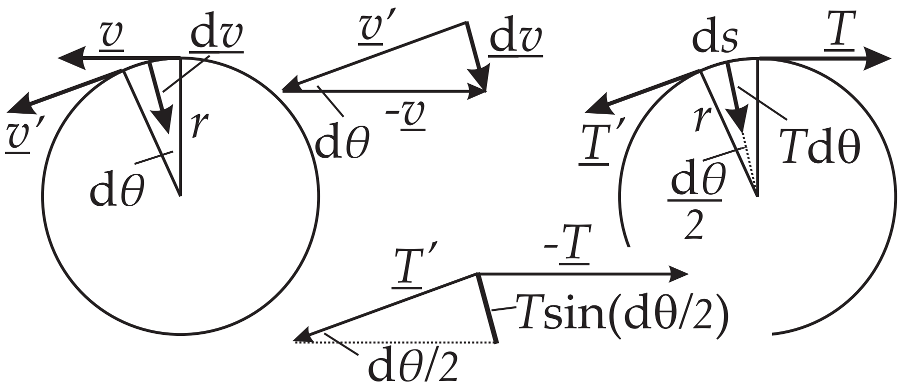

Figure 5: Centripetal acceleration for motion in a circle; inward forces from tangential tensions in a string on a curved path.

are equal in magnitude but not quite opposite in direction. Their difference $\mathrm{d}T = T(s + \mathrm{d}s) - T(s)$ points inward, see the vector triangle of figure 5 and is of magnitude $T\,\mathrm{d}\theta$; think of $\mathrm{d}T$ as a small arc of a circle of radius T. But from the actual circle $\mathrm{d}s = r\,\mathrm{d}\theta$, or $\mathrm{d}\theta = \mathrm{d}s/r$ whereupon $\mathrm{d}T = T\,\mathrm{d}s/r$. The inward force per unit length (dividing by $\mathrm{d}s$) is T/r.

This force generates the centripetal acceleration:

$$\lambda v^2/r = T/r \quad \rightarrow \quad \lambda v^2 = T.$$

The strange consequence of $1/r$ cancelling on both sides is that, if $T = \lambda v^2$, then a rope moving along its own length can turn *any* corner (r has disappeared from our considerations). We could dispose of the pulley altogether; indeed any other curve in the rope is stable.

A research problem — the chain fountain analysed

** You now have the machinery to analyse a difficult problem with the astonishing result seen in figure 1. This section can be skipped at a first reading if you want to proceed to catenaries, bridges, arches etc.

Figure 6 shows a schematic of the fountain that serves as a first model to analyse the fountain. The chain in a pile on the table rises up a height h_2, whereupon it turns around and then drops a height $h_2 + h_1$ to the floor, so the net drop is h_1. Tensions are marked at several points in the chain. In the curved region we know that $T_C = \lambda v^2$. In saying this we neglect the effect of gravity in the curved region in adding to the centripetal acceleration. It amounts to an assumption that $r \ll h_2$ and $r \ll h_1$. To account for gravity at the very tip of the arch involves a catenary analysis, which is done below, and the shape is not semi-circular. The results are qualitatively the same. The tension at the floor is marked T_F at the floor. A first assumption is that $T = 0$ there — with no chain below there can be no weight putting the chain under tension, which for most chains is reasonable and one can check this is the case for Mould's chain. It is not true for all chains and in effect a smooth, inert table can pull down with T_F, which is a shocking result presented by Grewal *et al*; see `http://www.youtube.com/watch?v=i9gLi4pBgpk`.

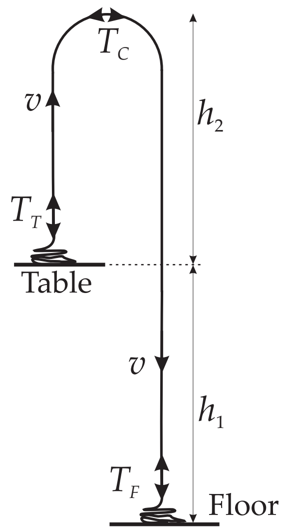

Figure 6: A model for the chain fountain of figure 1.

The chain fountain – 1 – naively, it does not exist!

The most obvious value for the tension at the table, T_T, is that required to accelerate the chain to speed v, that is $T_T = \lambda v^2$ which we proved is the force required to provide the momentum of the chain. We also have the relations between the tensions at different parts of the chain due to the dead weight of the sections between them. In summary:

$$T_C = \lambda v^2 \qquad T_C = T_T + \lambda h_2 g \qquad T_C = T_F + \lambda(h_2 + h_1)g$$

where we will take $T_F = 0$. One can immediately see that the first two of these equations are inconsistent with each other if $T_T = \lambda v^2$ unless also $h_2 = 0$. If the tension in the chain at the point of pickup is providing all the momentum to set the chain in motion, then there can be no chain fountain: $h_2 = 0$ means the chain just falls over the edge of the table without rising.

The chain fountain – 2 – it does exist, but something additional is pushing the chain at lift off!

Since Mould observed a fountain, then there must be a source of momentum additional to the tension at pickup, T_T, that is accelerating the chain. This can only be from the table pushing upwards with a force $R = \alpha\lambda v^2$. Here α is a constant (less than a $\frac{1}{2}$ as it turns out) and the rest of R is of the form λv^2 on dimensional grounds since the only function of such a force, were it to exist, is to accelerate the chain (the momentum flow of which is λv^2). That there is an upward force from the chain pile on the departing chain, is also shocking and is discussed below. For the moment one has to accept from momentum balance and the fact that fountains exist, there must be such a force. Thus the total force at pickup accelerating the chain breaks down as:

$$T_{\text{total}} = \lambda v^2 = T_T + \alpha\lambda v^2$$

whereupon the tension contributing to the acceleration at pickup is now rather $T_T = (1-\alpha)\lambda v^2$.

Exercise 5: Consider momentum initially imparted to the chain, and tension variation along the chain from the pick up point to the apex and then to the landing point. Show that the rise and speed of the chain is:

$$h_2 = h_1\alpha/(1-\alpha), \qquad v^2 = h_1 g/(1-\alpha).$$

Show also that the ratio of kinetic energy to potential energy change between the table and the floor is

$$KE/PE = \tfrac{1}{2}v^2/(gh_1) = \frac{1}{2(1-\alpha)}.$$

Thus place bounds on α and find the height of the highest fountain in terms of the drop h_1. Note that observed fountain heights for the Mould-type chains are $h_2 \sim 0.14 h_1$. [MW]

Why does the table push the chain at lift off?

Consider the physical origin of the reaction force $R = \alpha\lambda v^2$ needed for the chain fountain. Model the chain as a set of freely jointed rods each with length b, mass m and moment of inertia I. Links being set into motion are of necessity mostly at rest with a horizontal orientation, and are pulled at one end by preceding links that are directed largely upward. See figure 7 where a single link is picked up by a vertical force T_T applied at one of its ends. This upward force induces it to both rise and rotate. If the rod were in free space, the other end of the link would move down. When the rod sits on a horizontal surface (being the table or the rest of the pile of chain) then the pile supplies an additional upwards reaction force R to prevent this end of the link moving down. We pose a problem that gives an estimate of R in the initial phase of lift-up which we take as representative of R overall. Perhaps consult the RPP pages and problems on angular motion before attempting this question; see isaacphysics.org.

Exercise 6: Solve for the initial stages of the link's motion where there is a linear acceleration $\mathfrak{a}$ and an angular acceleration $\dot{\omega}$. As is common in maths and physics, $\dot{}$ denotes $\mathrm{d}/\mathrm{d}t$, the rate of change with time.

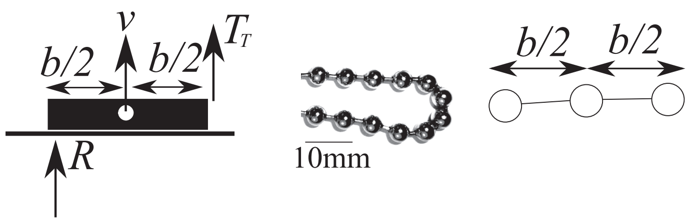

Figure 7: Left: A rigid rod of mass m and moment of inertia I lies on a horizontal surface (in practice the pile of chain) and is picked up via a vertical force T_T applied at one end causing the rod's center of mass to rise at a speed v. In order for the rod not to penetrate the surface, the surface must also provide a vertical reaction force R on the opposite end of the rod. Middle and Right: The ball chain in our experiments required 6 beads to turn by π (middle) so we model a link of the chain as consisting of 3 identical point masses (beads) connected by massless rods (right).

We use the initial force values as estimates of those acting during pickup.
Show that to avoid the end at R being rotated through the table, one needs $\mathfrak{a} = (b/2)\dot{\omega}$.
An R acts to achieve consistent rates of change of linear and angular momentum required to prevent the R-end rotating through the table. Show that

$$T_T + R = m\mathfrak{a} \quad \text{and} \quad (T_T - R)(b/2) = I\dot{\omega}.$$

From these equations and using $\dot{\omega} = \mathfrak{a}/(b/2)$, show that

$$R = \tfrac{1}{2}m\mathfrak{a}\left(1 - \frac{I}{\frac{1}{4}mb^2}\right).$$

Relating $m\mathfrak{a}$ to the forces and hence the upward momentum flow on pick up, show that

$$\alpha = \frac{R}{\lambda v^2} = \frac{1}{2}\left(1 - \frac{I}{\frac{1}{4}mb^2}\right).$$

This expression gives an upper limit on α that conserves energy; see the previous exercise where this issue is confronted. Estimate I of an effective link by recognizing that large bend in the chain is only achievable after traversing at least a few beads, figure 7. The number of beads required is an estimate of the length the equivalent rigid rod making up the chain. Find values of α and connect with observed h_2/h_1 values. [MW]

This analysis completes a simple analysis of the chain fountain. We now discuss the shape of static hanging chains, their relation to bridges and, in inverted form, to arches. One could then return to a detailed analysis of the chain fountain, which is an inverted catenary.

The shape of a hanging chain – 2

We derive the curve formed by the hanging chain, shown in figure 3.

Exercise 7: Recall from Ex. 2 that $\tan\phi(s) = \frac{s}{a}$ is the intrinsic form of a catenary curve for a hanging chain. Prove that with Cartesian coordinates x and y, along with the arc length s measured from the lowest point, one has

$$s/a = \sinh(x/a) \qquad \boxed{y(x)/a = \cosh(x/a)} \qquad y(s)/a = \sqrt{1 + (s/a)^2},$$

where the central $y(x)$ form is the Cartesian equation for a catenary. Draw $y(x)$, carefully specifying the origin of coordinates.

Hint: Prove that a small length along the chain $\delta s = \sqrt{(\delta x)^2 + (\delta y)^2} = \delta x\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}$. Explore the connection between this relation and $\phi(s)$. The triangle in Figure 8 shows a small section of the catenary. First prove the above connection between s and x. Substitutions, e.g. $u = \mathrm{d}y/\,\mathrm{d}x$, may be useful.

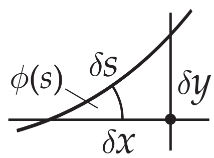

Figure 8: A small enough section of the curve $y(x)$ is approximately straight. Apply Pythagoras to the small triangle.

MW & PS.

A summary of forms and relations for a catenary; what it all means

The catenary has a shape given either as $y(x)$ or $\phi(s)$, or combinations of these variables. Also, the tension varies along the chain. We summarise the relations below. Test your understanding by reproducing and converting between them.

$$\begin{array}{rcc} s/a = \sinh(x/a) & y/a = \sqrt{1 + (s/a)^2} & y(x) = a\cosh(x/a) \\ \tan\phi(s) = s/a & y/a = \sec\phi(s) & a = T(0)/\lambda g \\ T(s) = T(0)\sec\phi(s) & T(y) = T(0)y/a & T(x) = T(0)\cosh(x/a) \\ T(s) = T(0)\sqrt{1 + (s/a)^2} & T(y) = \lambda g y & y(x) \geq a \end{array}$$

Explore the limits of these expressions and the role of the total length of chain, the separation of the end points, etc. Note that the angle $\phi(s)$ only approaches the vertical ($\phi \to \pi/2$) as $L/a \gg 1$ (locate the appropriate connection above) where $s = L$ is one end of the chain of total length $2L$. If the x coordinate of the right chain end is x_L and it approaches L, the chain is nearly stretched out tight. In that event, $\phi(L) \to 0$ and the above relations give $a \gg L$, both in the connection between ϕ and s and in the connection between s and x (check both). The tension is then enormous — the chain is very flat, supporting its weight with only a small component of its tension being in the vertical direction. See this from $T(0) = a\lambda g \gg \lambda L g$, the latter being the simple deadweight of the chain. Check the $T(s)$ relation then shows little variation with s.

A suspension bridge

Very strong, hanging cables support bridge carriageways by suspending cables between the strong major cable and the road. Now the mass per unit length λ is not from per unit length of the cable, but from per unit length of the load, i.e. the horizontal carriageway. This simplifies things greatly: the coordinate upon which angle and mass depends is simply x, the Cartesian coordinate, so $\phi = \phi(x)$ etc.

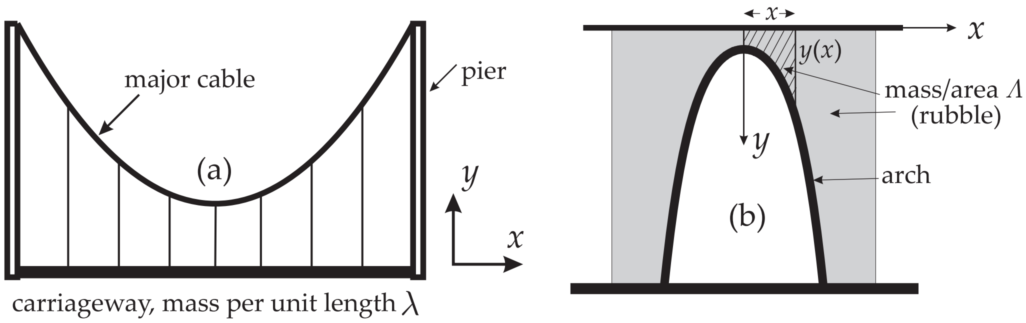

Figure 9: (a) A suspension bridge where the major cable supports a carriageway. (b) A Roman bridge in the form of a light arch supporting a rubble fill above it on which a carriageway sits.

Exercise 8: A suspension bridge is held up by light cables connected to a major cable supporting a carriageway of mass λ per unit length. It is supported by a pier at either end (see Figure 9(a)). Show that the major cable takes the shape $y = \frac{1}{2}\frac{x^2}{a}$ for suitable origins of x and y, and find an expression for a in terms of λ, g and the tension in the lowest part of the chain.

Hint: Consider the static equilibrium of a section of chain from $x = 0$ to x.

MW & CL

Arches; a Roman bridge

Recall that chain, ropes, threads etc. can only be in tension (no compression or shear/deflection). Likewise masonry can only be in compression. So masonry arches are like chains, with the signs of force & gravity reversed, and are analogies of the hanging structures we have examined above. So a pure arch, supporting nothing other than itself, would be an inverted catenary (prove this!). Arches that support things will need to be a different shape, just as we have seen that hanging ropes that support a bridge are not catenaries:

Exercise 9: A Roman method of bridge building was to form a masonry arch and fill in above it with rubble; figure 9(b). The arch is only stable if the lines of compressive stress follow the path of the arch. If the mass per unit area above the arch is Λ, show that the mass of rubble above the arch between $x = 0$ and x is $\int_0^x \Lambda y(x')dx'$. By considering the equilibrium of the *light* arch supporting this section of bridge, show that $\tan(\phi(x)) = \frac{\Lambda g}{T(0)} \int_0^x y(x')dx'$ where $T(0)$ is the compressive tension in the highest part of the arch, and $\phi(x)$ is the angle between the tangent to the arch at x and the x axis. Show further that the shape required for the supporting arch must be of the form $y = y_0 \cosh(x/a)$. Give an expression for a, and discuss the meaning of y_0. [MW & CL]

[MW & JSB with help from JC-Z & PS; Oct. 2013]